Extension 1 Mathematics
Preliminary Syllabus

Differentiation (A)

Specialist Homework Handout

Volume I

Student Name: ______________________________

Class: ______________________________

Term, Week: ______________________________

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1 Limits

- - - Stage 1 - - -

Question 1

Simplify the following limits.

(a) $\lim_{x\to 1} 3x-2$

(b) $\lim_{h\to 0} 2x+1+h$

Question 2

(a) Evaluate the following, given

$$g(x)=\begin{cases} x+2 & x<-2, \\ -x(x+4) & -2\le x\le 1, \\ x-6 & x>1. \end{cases}$$

(i) $\lim_{x\to 2^-} g(x)$

(ii) $\lim_{x\to 2^+} g(x)$

(iii) $\lim_{x\to -2} g(x)$

(iv) $\lim_{x\to 1^-} g(x)$

(v) $\lim_{x\to 1^+} g(x)$

(vi) $\lim_{x\to 1} g(x)$

(b) What can we conclude about the continuity at the point $x=-2$ and $x=1$?

Question 3

Evaluate

(a) $\lim_{x\to\infty} \frac{2}{x}$

(b) $\lim_{x\to\infty} \frac{3x+1}{x^2+x+1}$

(c) $\lim_{x\to\infty} \frac{2x^2+3}{3x^2+x+2}$

Question 4

Evaluate

(a) $\lim_{x \to 0} \frac{2}{x}$

(b) $\lim_{x \to 1} \frac{1}{x-1}$

(c) $\lim_{x \to \frac{1}{2}} \frac{x-1}{4x^2-1}$

Question 5

Evaluate

(a) $\lim_{x \to 4} \frac{x^2-16}{x-4}$

(b) $\lim_{x \to \sqrt{a}} \frac{x^2-a}{x-\sqrt{a}}$

(c) $\lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

(d) $\lim_{x \to 1} \frac{x^3-1}{x^2+x+1}$

(e) $\lim_{x \to -2} \frac{x+2}{3-\sqrt{x+11}}$

- - - Stage 2 - - -

Question 6

Evaluate

(a) $\lim_{x\to\infty} 2-\frac{1}{x+1}$

(b) $\lim_{x\to\infty} \frac{x-2}{x^2+x}$

(c) $\lim_{x\to\infty} \frac{4x^2+3}{x-2}$

(d) $\lim_{x\to\infty} \frac{1-x-x^2}{2+x+x^2}$

Question 7

Evaluate

(a) $\lim_{x\to-3} 2+\frac{1}{x+3}$

(b) $\lim_{x\to-1} \frac{x^2+1}{x^2-1}$

(c) $\lim_{x\to\frac{1}{3}} 1-\frac{2x}{9x^2-1}$

Question 8

Evaluate

(a) $\lim\limits_{x \to \sqrt{3}} \frac{\sqrt{3}-x}{3-x^2}$

(b) $\lim\limits_{x \to -2} \frac{x^3+8}{x^2+3x+2}$

(c) $\lim\limits_{x \to 3} \frac{x-3}{1-\sqrt{x-2}}$

Question 9

Evaluate

(a) $\lim\limits_{x \to -1} 3 + \frac{h}{x}$

(b) $\lim\limits_{x \to 1} \frac{x^2-x}{x^2-4x+3}$

(c) $\lim_{x \to a} \frac{\sqrt{2x+3}-\sqrt{2a+3}}{x-a}$

(d) $\lim_{x \to a} \frac{2x^2+6x-3a-ax}{x+3}$

Question 10

Evaluate the following given

$$f(x) = \begin{cases} -x & x < -1 \\ x^2 + 2x & -1 \le x \le 1 \\ \dfrac{3}{x} & x > 1. \end{cases}$$

(a) $\lim_{x \to -1} f(x)$

(b) $\lim_{x \to -1^-} f(x)$

(c) $\lim_{x \to -1+} f(x)$

(d) $\lim_{x \to 0^+} f(x)$

- - - Stage 3 - - -

Question 11

(a) Factorise $a^3 - b^3$.
(b) Hence show that

$$\lim_{x \to 0} \frac{(2+x)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x} = \frac{\sqrt[3]{2}}{6}.$$

Question 12

Given that $\lim_{x\to 0} \frac{ax^2+bx}{2x^2-4x} = 2$ and $\lim_{x\to\infty} \frac{ax^2+bx}{2x^2-4x} - 3$, find the values of a and b.

- - - Challenge Questions (Optional) - - -

Question 13

(a) Show that

$$\frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{1}{1-x^{2^n}} - \frac{1}{1-x^{2^{n+1}}}.$$

(Note that $x^{2^n} = x^{(2^n)}$).

(b) Using the result from (a), show that

$$\sum_{n=0}^{N} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{1}{1-x} - \frac{1}{1-x^{2^{N+1}}}.$$

(c) Let x be a real number with $-1 < x < 1$. Given that

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}},$$

show that

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{x}{1-x}.$$

(d) Hence, find

$$\sum_{n=0}^{\infty} \frac{1}{2014^{2^n} - 2014^{-2^n}}.$$

2 First Principles

- - - Stage 1 - - -

Question 1

Using first principles, find $\frac{dy}{dx}$ for the following.

(a) $f(x) = 4x^2$

(b) $f(x) = \sqrt{x+1}$

- - - Stage 2 - - -

Question 2

Differentiate the following using first principles.

(a) $y = 2x^2 + 3x - 5$

(b) $f(x) = \frac{2}{x^2}$

(c) $f(x) = x^3$ at $x = 2$

(d) $f(x) = \frac{x+2}{x-3}$ at $x = 1$

3 Rules of Differentiation

- - - Stage 1 - - -

Differentiation all questions in this set.

Question 1

(a) $y = 12$

(b) $y = 3x$

(c) $y = \frac{2}{x}$

(d) $y = 7\sqrt{x}$

(e) $y = \frac{1}{\sqrt{x}}$

(f) $y = \frac{1}{\sqrt{x^3}}$

(g) $y = \frac{\sqrt{x}}{15x}$

Question 2

(a) $y = x^2 - x - 1$

(b) $y = 2x^3 + \frac{1}{x}$

(c) $y = \frac{(x+1)^2}{5}$

(d) $y = 15x(x - 1)$

Question 3

(a) $y = (1 - x)^3$

(b) $y = (2x^2 - x + 3)^3$

(c) $y = \frac{7(x+1)^5}{5}$

(d) $y = \frac{1}{6(3x-1)}$

(e) $y = \frac{1}{5(x^2+2)^3}$

(f) $y = \sqrt{(5x^3 - 3x + 2)^3}$

Question 4

(a) $y = \sqrt{2x+1}$

(b) $y = \sqrt{x(x+2)}$

(c) $y = \sqrt{1 - \frac{2}{x}}$

(d) $y = 11\sqrt{1 - \frac{\sqrt{x}}{10}}$

Question 5

(a) $y = (3x^2 + x - 1)(x^2 + x + 2)$

(b) $y = 15x\sqrt{2x - 7}$

(c) $y = (1 - x)^3(3 + x)^2$

Question 6

(a) $y = \frac{2x-3}{3x+5}$

(b) $y = \frac{\sqrt{x}}{\sqrt{x}+1}$

(c) $y = \frac{3x+1}{\sqrt{1-x}}$

(d) $y = \frac{3x}{(x+1)^3}$

- - - Stage 2 - - -

Question 7

(a) $y = 5x - \sqrt{x^3} - \frac{2}{x}$

(b) $y = \frac{x^3}{3} - \frac{x^2}{2} - \frac{2}{\sqrt{x}}$

(c) $y = \frac{3(x-3)(x+3)}{2}$

(d) $y = \frac{7(2x-3)(2x+3)}{3x}$

Question 8

(a) $y = \frac{(ax^2+bx+c)^7}{7}$

(b) $y = \frac{(p-qx^2)^2}{q}$

(c) $y = \frac{5}{\sqrt{(x+1)^5}}$

(d) $y = -\frac{2}{\left(1-\sqrt{x}\right)^4}$

Question 9

(a) $y = \sqrt{3-(x+1)^2}$

(b) $y = \sqrt{x(x^2+1)}$

(c) $y = \sqrt{\frac{x^2+1}{x}}$

(d) $y = 8\sqrt{\frac{a}{\sqrt{x}}+b}$

Question 10

(a) $y = 21x\sqrt{3-x}$

(b) $y = (7+x)^2(3-2x)^2$

(c) $y = \sqrt{(x^2+1)(x^3+1)}$

(d) $y = \frac{1}{x\sqrt{x-2}}$

Question 11

(a) $y = \left(\frac{ax+b}{cx+d}\right)^n$ (b) $y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$ (c) $y = \frac{\sqrt{2x+1}}{x+2}$

- - - Stage 3 - - -

Question 12

Using the notation $\frac{d}{dx}f(x) = f'(x)$ and $\frac{d}{dx}g(x) = g'(x)$, express the following in simplest form.

(a) $\frac{d}{dx}\big(f(x)\big)^5$ (b) $\frac{d}{dx}\frac{1}{f(x)}$ (c) $\frac{d}{dx}f\big(g(x)\big)$

(d) $\frac{d}{dx}\left(f(x)+g(x)\right)$

(e) $\frac{d}{dx}\left(f(x).g(x)\right)^2$

(f) $\frac{d}{dx}\sqrt{\frac{f(x)}{g(x)}}$

- - - Mixed Practice - - -

Question 13

Differentiate the following.

(a) $y = 10x^3 + \frac{10}{x^3}$

(b) $y = (7x+1)\sqrt{x-2}$

(c) $y = \left(\frac{x-1}{x+1}\right)^2$

(d) $y = \frac{1+\sqrt{x}}{1-\sqrt{x}}$

(e) $y = \sqrt{x^2 + \sqrt{x}}$

(f) $y = \frac{\sqrt{2x-2}}{4x^2}$

(g) $y = \sqrt{\frac{3x+5}{3x}}$

(h) $y = \frac{1}{1-\sqrt{x}}$

(i) $y = -\frac{3}{\sqrt{x^2+1}}$

(j) $y = x\sqrt{x^2 - 3}$

(k) $y = \frac{5x+2}{\sqrt{8x+3}}$

(l) $y = x(x + 4)^7$

- - - Challenge Questions (Optional) - - -

Question 14

What is

$$\lim_{h\to 0}\frac{8\left(\frac{1}{2}+h\right)^8-8\left(\frac{1}{2}\right)^8}{h}?$$

Question 15

Show that

$$\lim_{x\to 5}\frac{x^3-x^2-100}{x-5}=65.$$

4 Tangent & Normal

- - - Stage 1 - - -

Question 1

Find the equation of the tangent to the curve $y = (2x - 3)^6$ at the point $x = 1$.

Question 2

Find the value of a if the gradient to the curve $y = ax^2 + x + 1$ at $x = 1$ is 3.

Question 3

Consider the curve $y = (x + 1)^2 - 3$. Find the point where the tangent is parallel to the line $3x - y = 1$.

Question 4

Find the equation of the normal to the curve $y = \frac{x}{(2x-3)^3}$ at the point $(1, -1)$.

Question 5

The tangent at the point P$(2, -4)$ to the curve $y = ax^2 + bx + 4$ is parallel to the line $y = 2x$. Find the values of a and b.

Question 6

Find the equation of the normal to the curve $y = \frac{x^2}{3}$ which is parallel to $4x + 2y - 1 = 0$.

- - - Stage 2 - - -

Question 7

The tangents of the parabola $y = x^2 + ax - 3$ at $x = 0$ and $x = 1$ are known to be perpendicular. Find the value of a.

Question 8

(a) Find the equation of the normal to $y = x^3 - 2x^2 - 3x + 1$ at P$(2, -5)$.

(b) Show that there is another point on the curve where the normal to the curve is parallel to the normal at P. Find the coordinates of this second point.

Question 9

Consider the curve $y = (x-1)^2 + 3$. Find the point where the tangent is perpendicular to the line $x + y - 1 = 0$.

Question 10

The normal to the curve $y = \frac{ax+b}{\sqrt{x}}$ has equation $4x + y = 22$ at the point where $x = 4$. Find the values a and b.

Question 11

Consider the curve $y = 4x^2(1 - x)$.

(a) Find the equations of the tangent and normal to the curve at that point $(1,0)$.

(b) The tangent and normal cut the y −axis at A and B respectively. If the point of intersection of the tangent and the normal is C find the area of $\Delta\mathrm{ABC}$.

- - - Stage 3 - - -

Question 12

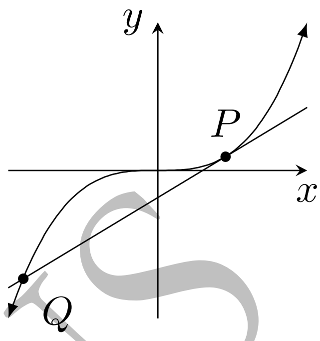
(a) Prove that the equation of the tangent at the point (x_0, y_0) on $y = x^2$ is $y = 2x_0x - x_0^2$.
(b) Find the equations of the tangent to the curve $y = x^2$ which are drawn from $(0, -1)$.
(c) Hence find where the tangent touches the curve given that the tangent passes through $(0, -1)$.

Question 13

(a) Prove that the curve $y = (x - 2)(x^2 + 2x + 6)$ crosses to the x −axis at only one point.
(b) Hence find the equation of the tangent at that point.

Question 14

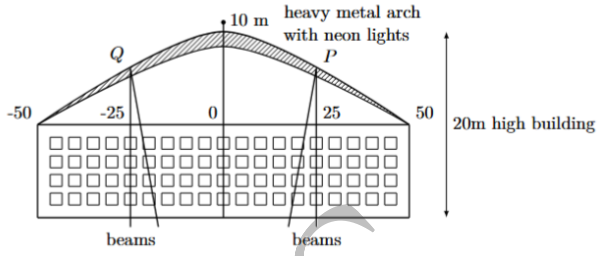
Let be a point on the curve $y = x^3$ and suppose the tangent line at P intersects the curve again at Q. Prove that the slope at Q is four times the slope at P.



Question 15

Metal arch with neon lights is to be fixed on top of a building as shown, the arch is in shape of a parabola with equation $y = -\frac{x^2}{250} + 10$.

The dimensions of the building are 100 m wide and 200 m high. Four beams are to be fixed at points P and Q to support the heavy metal arch i.e., two beams on each side of the building.



There is a beam at P and another at Q which are parallel to the y −axis of the parabola. Then there is another beam at iP as well as Q which forms normal to the curve at points P and Q. Given that the vertical beams are at a distance of 25 m from each side of the building, find the total length of the four support beams.

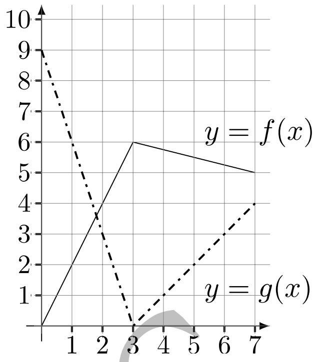
- - - Challenge Questions (Optional) - - -

Question 16

Let f and g be functions where $f'(2) = 2, g(2) = 1, f'(1) = 3$ and $g'(2) = -2$. What is the gradient of the tangent to the curve $y = f(g(x))$ at the point where $x = 2$?

Question 17

If $f(x)$ and $g(x)$ are the function whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(g(x))$.



(a) Evaluate $u'(1)$.
(b) Evaluate $v'(1)$.

Question 18

If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

Answers

1 Limits

1.
- (a) 1
- (b) $2x+1$

2.
- (a)
  - (i) 0
  - (ii) 4
  - (iii) 4
  - (iv) -5
  - (v) -5
  - (vi) -5
- (b) Discontinuous at $x = -2$, continuous at $x = 1$.

3.
- (a) 0
- (b) 0
- (c) $\frac{2}{3}$

4.
- (a) ∞
- (b) ∞
- (c) $-\infty$

5.
- (a) 8
- (b) $2\sqrt{a}$
- (c) $\frac{1}{2\sqrt{x}}$
- (d) 0
- (e) -6

6.
- (a) 2
- (b) 0
- (c) ∞
- (d) -1

7.
- (a) $-\infty$
- (b) ∞
- (c) $-\infty$

8.
- (a) $\frac{1}{2\sqrt{3}}$
- (b) -12
- (c) -2

9.
- (a) $3-h$
- (b) $-\frac{1}{2}$
- (c) $\frac{1}{\sqrt{2a+3}}$
- (d) a

10.
- (a) -1
- (b) 1
- (c) -1
- (d) 0

11.
- (a) $(a-b)(a^2+ab+b^2)$
- (b) Proof

12. $a = 6, b = -8$

13.
- (a) Proof
- (b) Proof
- (c) Proof
- (d) $\frac{2013}{2014}$

2. First Principles

1.
- (a) $8x$
- (b) $\frac{1}{2\sqrt{x+1}}$

2.
- (a) $4x+3$
- (b) $-\frac{4}{x^3}$
- (c) 12
- (d) $-\frac{5}{4}$

3 Rule of Differentiation

1.
- (a) 0
- (b) 3
- (c) $-\frac{2}{x^2}$
- (d) $\frac{7}{2\sqrt{x}}$
- (e) $-\frac{1}{2x\sqrt{x}}$
- (f) $-\frac{3}{2x^2\sqrt{x}}$
- (g) $-\frac{1}{30x\sqrt{x}}$

2.
- (a) $2x-1$
- (b) $6x^2-\frac{1}{x^2}$
- (c) $\frac{2x+2}{5}$
- (d) $30x-15$

3.
- (a) $-3(1-x)^2$
- (b) $3(2x^2-x+3)^2(4x-1)$
- (c) $7(x+1)^4$
- (d) $-\frac{1}{2(3x-1)}$
- (e) $-\frac{6x}{5(x^2+2)^4}$
- (f) $\frac{9}{2}(5x^2-1)\sqrt{5x^2-3x+2}$

4.
- (a) $\frac{1}{\sqrt{2x+1}}$
- (b) $\frac{x+1}{\sqrt{x^2+2x}}$
- (c) $\frac{1}{x^2\sqrt{1-\frac{2}{x}}}$
- (d) $-\frac{11}{40\sqrt{1-\frac{\sqrt{x}}{10}}}$

5.
- (a) $(6x+1)(x^2+x+2)+(2x+1)(3x+x-1)$
- (b) $\frac{15(3x-7)}{\sqrt{2x-7}}$
- (c) $-(1-x)^2+(3+x)(7+5x)$

6.
- (a) $\frac{19}{(3x+5)^2}$
- (b) $\frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)^2}$
- (c) $\frac{7-3x}{2(1-x)\sqrt{1-x}}$
- (d) $\frac{3(1-2x)}{(x+1)^4}$

7.
- (a) $5-\frac{3\sqrt{x}}{2}+\frac{2}{x^2}$
- (b) $x^2-x+\frac{1}{x\sqrt{x}}$
- (c) $3x$
- (d) $\frac{7(4x^2+9)}{3x^2}$

8.
- (a) $(2ax+b)(ax^2+bx+c)^6$
- (b) $-4x(p-qx^2)$
- (c) $-\frac{25}{2\sqrt{(x+1)^7}}$
- (d) $-\frac{4}{\sqrt{x}\left(1-\sqrt{x}\right)^5}$

9.
- (a) $-\frac{x-1}{\sqrt{3-(x+1)^2}}$
- (b) $\frac{3x^2+1}{2\sqrt{x^3+x}}$
- (c) $\frac{x^2-1}{2x^2\sqrt{\frac{x^2+1}{1}}}$
- (d) $-\frac{2a}{x\sqrt{x}\sqrt{\frac{a}{\sqrt{x}}+b}}$

10.
- (a) $\frac{63(2-x)}{2\sqrt{3-x}}$
- (b) $-2(7+x)(3-2x)(4x+11)$
- (c) $\frac{5x^4+3x^2+2x}{2\sqrt{(x^2+1)(x^3+1)}}$
- (d) $-\frac{3x-4}{2x^2(x-2)\sqrt{x-2}}$

11.
- (a) $n\left(\frac{ax+b}{cx+d}\right)^{n-1}\cdot\frac{ad-bc}{(cx+d)^2}$
- (b) $\frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}$
- (c) $\frac{-x+1}{(x+2)^2\sqrt{2x+1}}$

12.
- (a) $5\left(f(x)\right)^4.f'(x)$
- (b) $-\frac{f'(x)}{\left(f(x)\right)^2}$
- (c) $f'\left(g(x)\right).g'(x)$
- (d) $f'(x)+g'(x)$
- (e) $2\left(f(x)+g(x)\right)\left(f'(x)g(x)+g'(x)f(x)\right)$
- (f) $\frac{f'(x)g(x)-g'(x)f(x)}{2\left(g(x)\right)^2\sqrt{\frac{f(x)}{g(x)}}}$

13.
- (a) $30x^2-\frac{30}{x^4}$
- (b) $\frac{3(7x-9)}{2\sqrt{x-2}}$
- (c) $\frac{4(x-1)}{(x+1)^3}$
- (d) $\frac{1}{\sqrt{x}\left(1-\sqrt{x}\right)^2}$
- (e) $\frac{4x\sqrt{x}+1}{4\sqrt{x}\sqrt{x^2+\sqrt{x}}}$
- (f) $\frac{-3x^2+4}{4x^3\sqrt{2x-2}}$
- (g) $-\frac{5}{6x^2\sqrt{\frac{3x+5}{3x}}}$
- (h) $\frac{1}{2\sqrt{x}\left(1-\sqrt{x}\right)^2}$
- (i) $\frac{3x}{\sqrt{(x^2+1)(x^2+1)}}$
- (j) $\frac{2x^2-3}{\sqrt{x^2-3}}$
- (k) $\frac{20x+7}{\sqrt{8x+3}(8x+3)}$
- (l) $4(x+4)^6(2x+1)$

14. $\frac{1}{2}$

15. Proof

4 Tangents & Normal

1. $y = -12x+13$
2. $a = 1$
3. $\left(\frac{1}{2}, -\frac{3}{4}\right)$
4. $y = \frac{1}{7}(x-8)$
5. $a = 3, b = -10$
6. $32x+16y-27 = 0$
7. $a = -1$
8.
   - (a) $y = -x-3$
   - (b) $\left(-\frac{2}{3}, \frac{49}{27}\right)$
9. $\left(\frac{3}{2}, \frac{13}{4}\right)$
10. $a = 2, b = 4$
11.
    - (a) tangent: $y = -4(x-1)$; normal: $y = \frac{1}{4}(x-1)$
    - (b) $\frac{17}{8}$ u^2
12.
    - (a) Proof
    - (b) $y = 2x-1, y = -2x-1$
    - (c) $(-1,1), (1,1)$
13.
    - (a) Proof
    - (b) $y = 14(x-2)$
14. Proof
15. ***
16. $\frac{d}{dx}f\left(g(x)\right) = -6$
17.
    - (a) 6
    - (b) $\frac{3}{4}$
18. $f(4) \geq 16$